Supersymmetric Fokker-Planck strict isospectrality

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(Received 7 October 1996)

I report a study of the nonstationary one-dimensional Fokker-Planck solutions by means of the strictly isospectral method of supersymmetric quantum mechanics. The main conclusion is that this technique can lead to a space-dependent (modulational) damping of the spatial part of the nonstationary Fokker-Planck solutions, which I call strictly isospectral damping. At the same time, using an additive decomposition of the nonstationary solutions suggested by the strictly isospectral procedure and by an argument of Englefield [J. Stat. Phys. **52**, 369 (1988)], they can be normalized and thus turned into physical solutions, i.e., Fokker-Planck probability densities. There might be applications to many physical processes during their transient period. [S1063-651X(97)00808-8]

PACS number(s): 05.40.+j, 11.30.Pb

At present, supersymmetry is recognized as a basic symmetry of the natural world being discussed with various degrees of sophistication in many research areas. We also know that in physics it happens quite often that simple mathematical procedures might be extremely efficient and may lead to significant progress in clarifying our vision of the world. This is undoubtedly the case of Witten's supersymmetric quantum mechanics [1] which, mathematically speaking, is the factorization of the one-dimensional (1D) Schrödinger operator and relies on the Darboux transformations. The procedure involves Riccati equations for the so-called superpotential and most people became accustomed to working with the particular Riccati solution. However, Mielnik [2] realized that one can use the general Riccati solution and presented the 1D harmonic oscillator in that perspective, hinting of the connection with the inverse scattering methods. A clear discussion was provided by Nieto [3] and later Sukumar [4] showed that the Gel'fand-Levitan method can be interpreted as a sequence of two Darboux transformations. While the particular superpotential leads to the quasi-isospectrality of the partner Hamiltonians, the general one allows for the introduction of entire strictly isospectral families of either ''boson'' or ''fermion'' partner systems. In this paper I shall call Darboux-Witten (DW) isospectrality this strict isospectrality, which is obtained by employing the general superpotential (other people call it the double Darboux method). In a recent paper, Sukhatme and collaborators [5] obtained bound states in the continuum by means of DW isospectrality and a couple of other applications may be found in the literature [6]. In the following, after quickly presenting the factorization of the 1D Fokker-Planck (FP) equation, I shall tackle its DW (strict) isospectrality. Before proceeding, I recall that Darboux transformations (not DW ones) have been used by Englefield [7] for a FP equation with a particular potential, while Hron and Razavy [8] studied some solvable models of the FP equation by means of the Gel'fand-Levitan method. The other two techniques similar to the DW isospectrality, namely, the Abraham-Moses procedure [9] and the Pursey one [10], can be applied to the FP equation without any difficulty.

Bernstein and Brown [11] provided a simple discussion of the correspondence between the 1D FP equation with an arbitrary potential and Witten's supersymmetric quantum mechanics. They showed that the great advantage of the supersymmetric procedure for the FP problem is to replace bistable ''bosonic'' potentials with much simpler single well ''fermionic'' ones. Here, I shall use the Smoluchowski form of the 1D FP equation with constant diffusion coefficient and potential drift

$$\frac{\partial}{\partial t}\mathcal{P}(x,t)=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}+\gamma f'(x)\right]\mathcal{P}(x,t), \tag{1}$$

where $f'=df/dx$ is the drift force up to a sign, i.e., f is the FP potential, and γ is a free parameter whose expression depends on the physical problem under consideration. Equation (1) is not a Hamiltonian evolution of the solutions. Nevertheless it can be cast into the Schrödinger equation as follows. Any initial time-dependent solution $\mathcal{P}(x,t)$ will relax at asymptotic times to the stationary solution

$$\mathcal{P}_0(x)=\exp[-\gamma f(x)], \tag{2}$$

but to pass to probability densities $\mathbf{P}$ one should introduce a normalization constant N_C^{-1} as a factor in the right hand side of Eq. (2). The issue of normalization looks obvious but let me emphasize that to make it clear one should consider the FP probability current. This important quantity occurs when one wants to turn the FP equation into a continuity one. It reads

$$J(x,t)=-e^{-\gamma f}[e^{\gamma f}\mathbf{P}]'. \tag{3}$$

The stationarity is defined as the case(s) of constant J, and the constant is determined by the boundary conditions. The particular case of zero J means no periodic boundary conditions and leads to $e^{\gamma f}\mathbf{P}_0=C$, i.e., the constant is $C=1/N_C$. To get the common notion of probabilities (a set of positive numbers, each smaller than unity, summing up to unity, and

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mapped on an equivalent set of future, possible events belonging to the same class) one should take the normalization constant as follows:

$$N_C = \int e^{-\gamma f} dx, \tag{4}$$

where the integration limits are from zero to ∞ and from $-\infty$ to ∞ for the half-line and the full-line cases, respectively.

The evolution at intermediate times can be discussed conveniently by means of the celebrated ansatz

$$\mathcal{P}(x,t) = \psi(x,t)\exp\left(-\frac{1}{2}\gamma f(x)\right) = \psi(x,t)\sqrt{\mathcal{P}_0(x)}. \tag{5}$$

$\mathcal{P}(x,t) \to \mathcal{P}_0$ when $t\to\infty$, that is, $\psi(x,t) \propto \sqrt{\mathcal{P}_0(x)}$ at asymptotic times. The ansatz (5) turns the FP evolution of $\mathcal{P}$ into a Schrödinger evolution for the amplitude function ψ in imaginary time

$$\frac{\partial \psi}{\partial t} = -H_{\rm FP}\psi, \tag{6}$$

where the FP Hamiltonian is a Hermitian and positive semidefinite operator. It is now easy to proceed with the factorization and the whole Witten scheme for the FP Hamiltonian. We write $H_{\rm FP,1} = A_1 A_2$, with $A_1 = e^{\gamma f/2}(\partial/\partial x)e^{-\gamma f/2} \equiv \partial/\partial x - \frac{1}{2}\gamma f'$ and $A_2 = e^{-\gamma f/2}(\partial/\partial x)e^{\gamma f/2} \equiv \partial/\partial x + \frac{1}{2}\gamma f'$. I recall that the usual quantum-mechanical factorization is of the sort $H_{qm} = AA^\dagger + \epsilon$, where ϵ is the so-called factorization energy, which is a parameter fixing the energy scale. A zero value of this parameter, as in the case of the FP equation, indicates that the factoring of the Hamiltonian is done with respect to the "ground state" wave function. One can easily see that the FP superpotential is $\mathcal{W}_0 = (d/dx)\ln\sqrt{\mathcal{P}_0}$ and thus the mentioned FP factorization is indeed with respect to the FP ground state amplitude. Also, I remark that the FP superpotential $\mathcal{W}_0 = -1/2\gamma f'$ is proportional to the drift force, which is a quite well-known result. The superpartner Hamiltonian will be $H_{\rm FP,2} = A_2 A_1$. The two FP Hamiltonian partners can be written as

$$-H_{\rm FP,1,2} = \frac{d^2}{dx^2} - S_{1,2}, \tag{7}$$

with the Schrödinger potentials $S_{1,2}$ entering simple Riccati equations

$$S_{1,2} = e^{\pm\gamma f/2}\left(\frac{d^2}{dx^2}e^{\mp\gamma f/2}\right) = \left(\frac{\gamma f'}{2}\right)^2 \mp \frac{\gamma f''}{2}. \tag{8}$$

At this point, it is worthwhile to emphasize the connection between the FP solutions and the Schrödinger solutions. According to Eq. (5), the FP solutions can always be decomposed into a product of a nonstationary Schrödinger solution and the ground state amplitude of the corresponding stationary Schrödinger equation. As remarked by Englefield [7], $\sqrt{\mathcal{P}_0}$ in Eq. (5) can be any positive solution of the stationary Schrödinger equation, i.e., does not have to be normalizable. This is a crucial point for our arguments below.

Let us pass now to the strictly isospectral construction. In principle, one can use any solution $\varphi(x)$ of the *stationary* Schrödinger equation to perform this construction [5,6], but we shall use the ground state amplitude $\varphi_0(x) = \sqrt{\mathcal{P}_0}$. It is saying that the strictly isospectral Schrödinger potentials are given by

$$S_{\rm iso,1}^{\rm DW}(x,\varphi_0;\lambda) = S_1(x) - 2[\ln(\mathcal{J}+\lambda)]'' = S_1(x) - \frac{4\varphi_0\varphi_0{}'}{\mathcal{J}+\lambda} + \frac{2\varphi_0^4}{(\mathcal{J}+\lambda)^2}, \tag{9}$$

where λ is a real parameter, that mathematically is the arbitrary integration constant of the general Riccati solution, and

$$\mathcal{J}(x) \equiv \int_c^x \varphi_0^2(y)\,dy, \tag{10}$$

where $c=0$ for the half-line case and $c=-\infty$ for the full-line one, whereas the expression of $\varphi_{\rm gen}$ is given below. In this construction the φ_0 solution is reintroduced in the spectrum by using the general superpotential solution of the Riccati equation which reads

$$\mathcal{W}_{\rm gen} = -\frac{1}{2}\gamma f' + \frac{d}{dx}\ln[\mathcal{J}(x)+\lambda]. \tag{11}$$

The way to obtain Eq. (11) is well known [2,3] and will not be repeated here. From it one can easily get Eq. (9). The new class of solutions is a one-parameter family $\varphi_{\rm gen}$ differing from φ_0, by a quotient

$$\varphi_{\rm gen}(x;\lambda) = \frac{\varphi_0(x)}{\mathcal{J}+\lambda} = \varphi_0(x)/M_D(x). \tag{12}$$

In the FP context we need $\varphi_{\rm gen} > 0$, i.e., $\lambda \in (-\mathcal{I}(x),\infty)$. The range of λ will be further restricted by a normalization condition (see below). The main point now is that the denominator $M_D(x)$ looks like a space-dependent modulation. Its general behavior has been obtained in [5,6]. There is a strong damping effect of the integral $\mathcal{J}$, both for the family of potentials and for the wave functions. The parameter λ is just signaling the importance of the integral term. There is some modulation close to the origin and for small λ parameters. At higher λ values the damping nature gets extremely strong and $\varphi_{\rm gen}$ is going rapidly to zero.

The general form of the FP amplitude solution using the general Riccati solution reads

$$\varphi_{\rm gen} = \exp\left(\int \mathcal{W}_{\rm gen}\right) = \exp\left(-\frac{1}{2}\gamma f\right)\Big/ M_D \tag{13}$$

and thus one can see the self-modulation of the solutions. These modulated solutions are stationary solutions of the Schrödinger equation

$$\frac{d^2\varphi(x)}{dx^2} - S_{\rm iso,1}^{\rm DW}(x,\varphi_0;\lambda)\varphi(x) = 0, \tag{14}$$

corresponding to FP equations which are strictly isospectral to the initial one. Exactly as in supersymmetric quantum mechanics, the strictly isospectral FP equations can be understood as a one-parameter family of ''bosonic'' FP equations having the same ''fermionic'' partner equation. Its factoring operators are $B_1=\partial/\partial x+\mathcal{W}_{\rm gen}$ and $B_2=\partial/\partial x-\mathcal{W}_{\rm gen}$. The amplitude $\psi_1=B_1\psi$ satisfies the time-dependent Schrödinger equation (6) unless S_2 substitutes for S_1. Using now the essential fact that $1/\varphi_{\rm gen}$ (''ground state zero mode'') is the general solution of Eq. (6) again when S_2 replaces S_1 (see Sec. 7.1 in the review of Cooper, Khare, and Sukhatme [1]), and being positive can be employed in Eq. (5) instead of $\sqrt{\mathcal{P}_0}$ whenever ψ_1 substitutes for ψ, one gets the following FP solution

$$\mathcal{P}_1(x,t)=\varphi_{\rm gen}^{-1}(B_1\psi)=\frac{\partial}{\partial x}\left(\frac{\psi}{\varphi_{\rm gen}}\right). \tag{15}$$

If one calculates the normalization integral for this solution one gets

$$N_1=\int_{-\infty}^{+\infty}\mathcal{P}_1(x,t)dx=(\psi\varphi_{\rm gen}^{-1})|_{-\infty}^{+\infty}. \tag{16}$$

For confining FP potentials, i.e., going to ∞ at both asymptotic limits one has $1/\varphi_{\rm gen}\rightarrow 0$ for any λ and therefore N_1 is zero. On the other hand, $\varphi_{\rm gen}$ can be normalized and of course adding any multiple of $\mathcal{P}_1$ will not change the normalization constant. In this way, one arrives at strictly isospectral FP probabilities of the type

$$\mathbf{P}_2(x,t)=k\mathcal{P}_1(x,t)+N_{\rm gen}^2\varphi_{\rm gen}^2, \tag{17}$$

where the normalization constant $N_{\rm gen}$ is [12]

$$N_{\rm gen}=\int_{-\infty}^{+\infty}\varphi_{\rm gen}^2(x)dx=\sqrt{\lambda(\lambda+1)}. \tag{18}$$

The constant k is arbitrary, restricted by the physical condition $\mathbf{P}_2\geqslant 0$. Moreover, $N_{\rm gen}$ imposes further restrictions on the λ parameter. The correct range of λ in any physical problem related to FP transients should be $\lambda\in(-\mathcal{I}(x),-1)\cup(0,\infty)$.

The decomposition Eq. (17) of the transient FP probabilities appears to be quite general. The behavior of the transient probabilities is dictated by the parameter λ, which is also essential in the normalization issue. On the other hand, since $S_{\rm iso}^{\rm DW}\rightarrow S_1$ for $\lambda\rightarrow\infty$, the decomposition Eq. (17) no longer applies in that limit, and one should use the common decomposition Eq. (5). In other words, although the isospectral solutions are stationary ones they are related to the transient period and disappear in the asymptotic stationary regime. The parameter λ looks like an effective ''time'' (damping) parameter measuring their importance during the transient phases.

Perhaps it is worthwhile to remark that in our notations the initial FP equation reads

$$\frac{\partial}{\partial t}\mathcal{P}(x,t)=\frac{\partial}{\partial x}\left[A_2+\frac{1}{2}\gamma f'\right]\mathcal{P}(x,t). \tag{19}$$

Substituting A_2 by A_1 gives a FP form of the heat equation

$$\frac{\partial}{\partial t}\mathcal{P}(x,t)=\frac{\partial}{\partial x}\left[A_1+\frac{1}{2}\gamma f'\right]\mathcal{P}(x,t), \tag{20}$$

since the operator in the brackets is $\partial/\partial x$.

The strictly isospectral FP equations can be written by analogy as

$$\frac{\partial}{\partial t}\mathcal{P}(x,t)=\frac{\partial}{\partial x}\left[B_2+\frac{1}{2}\gamma f'\right]\mathcal{P}(x,t) \tag{21}$$

and the corresponding isospectral ''heat'' equation will be

$$\frac{\partial}{\partial t}\mathcal{P}(x,t)=\frac{\partial}{\partial x}\left[B_1+\frac{1}{2}\gamma f'\right]\mathcal{P}(x,t). \tag{22}$$

Finally, I recall that besides the DW strictly isospectral potential Eq. (9), there are four other distinct families of strictly isospectral Schrödinger potentials obtained from pair combinations of Darboux-Witten, Pursey, and Abraham-Moses procedures [13]. The formulas corresponding to Eq. (9) for the four families are collected in the paper of Khare and Sukhatme [13]. All of them are of the Darboux type and one can use any of them in the FP problem. From the scattering point of view all these families have the same reflection and transmission probabilities. However, they differ from each other in reflection and transmission amplitudes.

In conclusion, I have shown in a simple and general manner that DW strict isospectrality may provide interesting insights in the case of FP probabilities.

This work was supported in part by the Consejo Nacional de Ciencia y Tecnología (Mexico) Project No. 4868-E9406.

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